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Report 68-1

UNDER-ICE ACOUSTIC STABILITY: PRELIMINARY EXPERIMENT AND THEORY OF MEASUREMENT (U)

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UNDER-ICE ACOUSTIC STABILITY: PRELIMINARY EXPERIMENT AND THEORY OF MEASUREMENT

A B S T R A C T

Sound-paths under sea-ice appear to be more stable than those in ice-free seas. High stability may permit a certain degree of reverberation cancellation to be utilized in fixed location echo ranging. The results of a preliminary experiment which measured the stability of direct sound-paths under the sea-ice of M'Clure Strait in the Canadian Arctic Archipelago are described.

A theoretical method is developed for the measurement of the range dependent stability of sound-paths for reverberation under sea-ice. The ratio of the coherent to the total sound energy in a sequence of range-gated reverberation returns is given as a criterion for stability.

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1.0 INTRODUCTION

Limitations to effective echo-ranging under Arctic sea-ice are a consequence of high reverberation levels produced by the rough under-side of the ice. Although high rms reverberation levels are generally observed, there is a significant difference in the character of the reverberation under sea-ice compared to ice-free seas. There appears to exist a significant correlation between the instantaneous reverberation noise following the transmission of a sound pulse and the reverberation noise observed one pulse-repetition period later. In ice-free seas, where surface waves are the major source of reverberation noise, the surface configuration changes during a pulse-repetition period. In fact, the changing sea surface produces a spectral broadening in reverberation*. For a fixed projector system under sea-ice this is not always the case, especially when the ice is shore-fast. If, in particular, the sea-ice is considered to be geometrically fixed, then correlation between consecutive reverberation pressure-time functions will be degraded only by the changes in the sound transmission properties of the water beneath the ice. In such a case a decrease in correlation should be observed in reverberation as time or range increases. In other words, sound-paths under sea-ice must be highly stable for any significant correlation to exist between successive reverberation pressure waves.

In order to test the hypothesis of high sound-path stability in a real Arctic environment, a reverberation and transmission experiment was planned and partially executed during April and May of 1966 in M'Clure Strait (see map in Figure 1).

Reverberation recordings were not obtained owing to the failure of a high power transducer. However, changes in phase-angles in direct sound transmission paths were measureable to within a resolution of $\pm 3^{o}$. Although phase-angle changes caused by the medium were not detectable with the resolution available, the stability appeared to be sufficient to support the hypothesis.

Subsequent to the field experiment, a theoretical method of stability measurement has been evolved. The bulk of this report is concerned with a method of measurement of range-dependent sound-path stability in reverberation under sea-ice and the relationship of this stability to the energy-spectrum of a sequence of hypothetical time-gated reverberation pressure-time signals.

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*For a theoretical treatment of spectral broadening in reverberation for ice-free seas see reference 1.

2.0 A TEST OF SOUND-PATH STABILITY IN M'CLURE STRAIT, 1966.

In the spring-time, M'Clure Strait is normally covered with rough pack-ice which is kept immobile by Melville and Banks Islands for the greater part of the year. Logistic support was available from the department of Energy, Mines and Resources at Mould Bay on nearby Prince Patrick Island. M'Clure Strait was therefore chosen as an appropriate location wherein to verify or disprove the concept of high underwater acoustic sound-path stability.

In the planning stages, the only verification of the existence of some stability was available from reverberation envelopes obtained in the Arctic Ocean in the previous spring. Backscattered sounds from explosive charges which were detonated under the sea-ice in approximately the same location, produced nearly identical reverberation envelopes. Therefore, by using pulsed-cw signals emitted and received by a fixed transducer, one might expect to observe the degree of stability for sound-paths in the medium by measuring the lack of identity between successive reverberation traces in both amplitude and phase. This lack of identity could then be attributed to the motion of the ice cover in addition to effects caused by changes in the speed of sound in the water.

The experimental equipment, transported to a field camp on M'Clure Strait, was arranged according to the diagram of Figure 2. The transmitting element was either a 4 kHz cylindrical unit for the frequency range 3-4 kHz, or a 1 kHz unit for frequencies in the 0.6-1.2 kHz range. Either of these elements was mounted on the bottom when required. The transducer circuitry included a "transmit-receive" switch to allow a transducer to be used as a hydrophone to receive the expected reverberation noise. Two other hydrophones were mounted through holes in the ice at horizontal distances of 402 m. and 805 m. for receiving directly transmitted pulses.

The plans for reverberation measurements were based on the power transmission capabilities of the 1 kHz transducer which, however, failed to survive temperature changes on transportation to the site. As a result, it could only be used at a much reduced power level for the direct transmissions of pulsed-cw signals to the two hydrophones supported under the ice.** The field work was therefore restricted to recording the two separate signals received by the surface hydrophones. One additional track in the magnetic tape recorder preserved a replica of the timing wave, from which the transmitted pulses were gated, while another preserved replicas of the transmitted pulses. The method of recording was to run off one

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** Direct and indirect transmissions of cw signals and their frequency smearing has been described by Ellenthorpe in ref.2 for ice-free seas.

complete reel of magnetic tape while transmitting pulses every 5 seconds. Each tape-reel preserved approximately ninety pairs of pulses over an 8 minute time interval.

A complete description of the analysis of the tape records is contained in reference 3. The essence of the conclusions are as follows:

(i) Over an 8 minute time interval significant phase changes in the direct sound-paths were observable only for recordings made at the higher frequencies, i.e. 4 kHz.

(ii) The phase shifts observed could be attributed to frequency drifts in the oscillator which generated the timing wave and from which the transmitted pulses were gated.

(iii) Uncertainty as to the exact frequency drift of the timing oscillator did not warrant the use of a more accurate means of phase measurement, which was restricted to a resolution limit of $\pm 3^{\circ}$. For the 805 m. sound-path and for transmissions made at 4 kHz, this resolution corresponded to a wavelength stability of $\pm$ 1 part in 3×10^{5}.

Subsequent to the above analyses, measurements were made on the phase errors introduced between identical 4 kHz cw waves recorded on two adjacent tracks on the magnetic tape recorder. These measurements showed that 95% of the phase angle errors were confined within the limits of $+ 2.8^{\circ}$.

The results showed that, under the conditions of this experiment, the short term stability (8 min.) appeared to be very high indeed and was not measureable with the apparatus used in the exploratory experiment in M'Clure Strait.

Plans have been laid for a reverberation experiment in April, 1968, using: (1) a more suitable transducer, (2) a precision frequency source and (3) an improved method of magnetic tape recording.

3.0 THEORETICAL MODEL OF A REVERBERATION EXPERIMENT

The reverberation noise under sea-ice can be considered to arise from scatterers which are positionally fixed in the medium. Temporal fluctuations in the transmission characteristics of the medium will then be equivalent to an apparent motion of the scatterers. Where no apparent motion of the scatterers exists, coherent backscattered waves will be observed. The backscattered waves will be coherent in the sense that identical reverberation time-wave forms will be observed for each identical pulse transmitted at zero time when the projector, receiver and scatterers are geometrically fixed. If each scatterer has an apparent random motion, independent of all other scatterers about its positional mean, there will exist an incoherent contribution to the back-scattered waves.

It is clear that coherence, or lack of it, can only be observed by making suitable comparisons of one reverberation pressure-wave with another, and that the statistical properties of the coherent content can only be assessed by observing many reverberations with respect to a common position and time base.

The coherent content of the reverberation can be identified as a known signal upon which a target signal may be superimposed. The incoherent reverberation is a non-stationary noise which, when added to system and ambient noise, will tend to mask the signals.

One therefore asks the questions: (1) What are the properties of coherent and incoherent reverberation energy expected in a particular under-ice environment? (2) How does this proportion vary with distance and with frequency? (3) What is the spectrum of the reverberation relative to its coherent and incoherent content? (4) Finally, what would be the expected phase or time-jitter for a single pulsed transmission to a particular range?

A single experiment can be conceived of to answer most of these questions and one could then infer the answers to the remainder. To simplify the electronics and the analysis, pulsed cw signals emitted and received by a common transducer are desireable. The problem of reverberation noise non-stationarity and of an assumed change in the proportion of coherent and incoherent reverberation with range dictates the need for analysis by range or time-gating of the reverberation. The range-gated reverberation can then be treated as a sample of stationary noise. Equipment simplicity and the requirement for statistical averaging over the echoes from many back-scattered waves are served by the use of a wide aperture transducer. The frequency dependence can then be inferred only by assuming that changes in the transmission paths in the water will be responsible for the reverberation incoherence and that the expected phase-jitter to a particular range will be a linear function of the frequency. The hypothesis which assumes that changes in the transmission paths in the water produce incoherent reverberation can be tested by the examination of all range-gates.

Referring to the particular geometry of Figure 3, which shows a projector-hydrophone having a toroidal radiation pattern and sitting just off the bottom, it can be assumed that the reverberation is caused by the rough ice-water interface and that a time-gated sample of reverberation corresponds to reverberation from scatterers within an annular area of width ΔR. Therefore, a single outgoing pressure-pulse, formed by gating a sinusoidal timing waveform, will result in a reverberation pressure wave shown in the sketch of Figure 4. A gated-interval of duration K seconds is shown and the time t_o is equal to the mean slant range r_o divided by the mean sound-speed c.

Each gated interval of reverberation of duration K seconds, will now be referred to an an "echo-set". If pulses are transmitted every T seconds for γT seconds, γ reverberation waveforms can be recorded as

sketched in Figure 5. By gating for K seconds centred on times t_o, $T + t_o$, -----$mT + t_o$, -----$\gamma T + t_o$, the echo-sets sketched in Figure 6 are obtained. Each echo-set can be considered as an environmental acoustic sample for a range interval ΔR.

At this point, it is intuitively evident that, if all the echo-sets have identical waveforms, then the environment has been acoustically "rigid", at least to the range r_o for γT seconds. Likewise, lack of "rigidity" will be displayed in the dissimilarities in the wave forms of the echo-sets.

The object of the following analysis is to relate the expected single-path time-jitter to the form of the energy spectrum of γ echo-sets.

For example, it will be shown (equation 17b) that the expected energy spectral density for γ independent echo-sets has two components; one component being similar to the spectral density for a single echo-set while the second is a comb spectrum. The characteristic function for a single-path time-jitter variable determines the division of energy between the two spectral components.

3.1 THE WAVEFORM OF AN ECHO-SET

At the outset, the reverberation from sea-ice has been assumed to correspond to the superposition of back-scattered wavelets from independent point scatterers. Each αth scatterer is assumed to have an independent cross-section which will result in a wavelet with pressure amplitude a_α/r_o^2 at the hydrophone. Secondary scattering is ignored and propagation by ray-paths is assumed to be valid.

Medium fluctuation can be interpreted as a random motion of each scatterer in the radial direction about its positional mean. With these assumptions, an outgoing pressure pulse $u(t_1-mT,\tau_o)$ will reappear from the αth scatterer as $a_\alpha r_o^{-2} u(t_1-t_{\alpha m}-mT,\tau_\alpha)$ where τ_o is the transmitted pulse length and $\tau_\alpha \leq \tau_o$ because of the possibility of truncation at the edge of the gated interval. The time $t_{\alpha m}$ is a random variable distributed uniformly in the gated time interval K. This form of the scattered wavelet is valid provided there is a negligible apparent acceleration of the scatterer during the transit time of a pulse; also, dependence on the echo-set number m will exist if the αth scatterer has an apparent oscillation with a frequency component which is comparable to or lower than the pulse repetition frequency (1/T).

Realizing that a narrow band-width pulse is being considered, we can write the pulse-waveform as

$$u(t_1) = g(t_1)e^{-j\omega_o t_1}, \text{ - - - -} \qquad (1)$$

where ω_o is the circular frequency of the timing waveform and $g(t_1)$ is an envelope function. Also, the gated reverberation echo-set can be written as:

$$k_m(t_1-mT) = r_o^{-2} \sum_{\alpha=1}^{n_o} a_\alpha u(t_1-t_{\alpha m}-mT,\tau_\alpha),$$

where n_o is the total number of equivalent scatterers in the echo-set. Consequently the echo-set waveform is:

$$k_m(t_1-mT) = r_o^{-2} \sum_{\alpha=1}^{n_o} a_\alpha g(t_1-t_{\alpha m}-mT,\tau_\alpha) e^{-j\omega_o(t_1-t_{\alpha m})} \qquad (2)$$

where $e^{-jmT\omega_o} = 1$.

3.2 THE AUTOCORRELATION FUNCTION FOR γ ECHO-SETS

The auto-correlation function for γ echo-sets can be expressed as the statistical expectation of their time-delayed product.

From (2), the waveform of γ echo-sets, shown in Figure 6 is:

$\sum_{m=o}^{\gamma-1} k_m(t_1-mT)$, so that the autocorrelation function is:

$$R(t) = E\left\{\sum_{m=o}^{\gamma-1} k_m^*(t_1-mT) \sum_{n=o}^{\gamma-1} k_n(t_1-nT+t)\right\}$$

$$= r_o^{-4} E\left\{\sum_{m=0}^{\gamma-1}\sum_{\alpha=1}^{n_o} a_\alpha g(t_1-t_{\alpha m}-mT,\tau_\alpha) e^{j\omega_o(t_1-t_{\alpha m})}\right.$$

$$\left.\times \sum_{n=o}^{\gamma-1}\sum_{\beta=1}^{n_o} a_\beta g(t_1-t_{\beta n}-nT+t,\tau_\beta) e^{-j\omega_o(t_1-t_{\beta n}+t)}\right\}. \qquad (3a)$$

Since $t_{\alpha m}$ and $t_{\beta n}$ are independent variables for $\alpha \neq \beta$, there is no joint probability for the existence of product terms for $\alpha \neq \beta$, consequently, in(3a), $\beta=\alpha$ only. R(t) can then be expanded into self products for m=n and cross products for m≠n. If we assume that the medium is statistically stationary over the time γT, then all the self-products will have an identical expectation with terms for m = n. Likewise there will be $(\gamma-1)$ terms for n = m+1 with identical expectations, - - - - - - - -$(\gamma-p)$ terms for n = m+p - - - - -, and finally one term for $n = m+(\gamma-1)$. Also, as a consequence of stationarity, the expectations will depend only on the difference integer, p = n-m, so that we can set m = o. R(t) from (3a) is now expressed as:

$$r_o^4 R(t) = \gamma E \left\{ \sum_{\alpha=1}^{n_o} a_\alpha^2 g(t_1 - t_{\alpha o}, \tau_\alpha) g(t_1 - t_{\alpha o} + t, \tau_\alpha) e^{-j\omega_o t} + - - - - \right.$$

$$+ (\gamma - p) E \left\{ \sum_{\alpha=1}^{n_o} a_\alpha^2 g(t_1 - t_{\alpha o}, \tau_\alpha) g(t_1 - t_{\alpha p} + t - pT, \tau_\alpha) e^{-j\omega_o (t + t_{\alpha o} - t_{\alpha p})} + - - - \right.$$

$$+ E \left\{ \sum_{\alpha=1}^{n_0} a_\alpha^2 g(t_1 - t_{\alpha o}, \tau_\alpha) g(t_1 - t_{\alpha, \gamma-1} + t - (\gamma-1)T) e^{-j\omega_o (t + t_{\alpha o} - t_{\alpha, \gamma-1})} \right.$$

$$= \sum_{p=o}^{\gamma-1} (\gamma - p) E \left\{ \sum_{\alpha=1}^{n_o} a_\alpha^2 g(t_1 - t_{\alpha o}, \tau_\alpha) g(t_1 - t_{\alpha p} + t - pT, \tau_\alpha) e^{-j\omega_o (t + t_{\alpha o} - t_{\alpha p})} \right\} \quad (3b)$$

Computation of the averages in (3b) require that the time γT is much larger than the maximum period of the significantly probable fluctuations of the medium. In other words, the random components in the time variables $t_{\alpha o}$ and $t_{\alpha p}$ must become independent for $p \ll \gamma$ for at least the majority of scatterers. Conversely, if the pulse repetition frequency $1/T$ is always lower than the lowest significant frequency in the spectrum of the apparent radial oscillation of the scatterers, than $t_{\alpha o}$ and $t_{\alpha p}$ are independent variables and then each echo-set is an independent sample of the acoustic environment.

3.3 THE TIME-JITTER VARIABLE

At this point it is expedient to express the time $t_{\alpha p}$ as the sum of a fixed mean-time or "ranging time" t_α and a "jitter-variable" $q_{\alpha p}$ so that

$$t_{\alpha p} = t_\alpha + q_{\alpha p}; \text{ hence:} \quad (4a)$$

$$t_{\alpha o} - t_{\alpha p} = q_{\alpha o} - q_{\alpha p}. \quad (4b)$$

Also assume the existence of a joint probability density function $p(q_{\alpha o}, q_{\alpha p})$ with the sensible property of being identical for all scatterers.

$$\text{Thus } p(q_{\alpha o}, q_{\alpha p}) = p(q_o, q_p). \quad (5a)$$

Likewise, the individual probability density functions for each scatterer are assumed to be identical for all α and all p so that:

$$p(q_{\alpha o}) = p(q_{\alpha p}) = p(q). \qquad (5b)$$

3.4 EXPANSION OF THE PULSE-ENVELOPE FUNCTION IN R(t)

The pulse envelope functions in (3b), as defined in (1), can be expanded in a Taylor's series. Also using the jitter-variable in (4) we obtain:

$$g(t_1-t_\alpha-q_{\alpha p},\tau_\alpha) \approx g(t_1-t_\alpha,\tau_\alpha)-q_{\alpha p}\dot{g}(t_1-t_\alpha,\tau_\alpha)$$

where $\dot{g}(x) = \frac{d}{dx}\, g(x)$.

Consequently from (3b):

$$g(t_1-t_{\alpha o},\tau_\alpha)g(t_1-t_{\alpha p}+t-pT,\tau_\alpha) \approx$$

$$g(t_1-t_\alpha,\tau_\alpha)g(t_1-t_\alpha+t-pT,\tau_\alpha)-q_{\alpha o}\dot{g}(t_1-t_\alpha,\tau_\alpha)g(t_1-t_\alpha+t-pT,\tau_\alpha)$$

$$-q_{\alpha p}\dot{g}(t_1-t_\alpha+t-pT,\tau_\alpha)g(t_1-t_\alpha,\tau_\alpha)+q_{\alpha o}q_{\alpha p}\dot{g}(t_1-t_\alpha,\tau_\alpha)\dot{g}(t_1-t_\alpha+t-pT,\tau_\alpha). \qquad (6)$$

3.5 R(t) EXPRESSED IN TERMS OF PROBABILITY DENSITY FUNCTIONS

The expectations with respect to the amplitudes a_α can be expressed in terms of a common probability density function $p(a)$, and since all scatterers are by definition independent, the expectation with respect to a_α^2 is then σ_a^2 where $\sigma_a^2 = \int a^2 p(a)da$. Also by using the fact that the expectation for the times t_1-t_α become, by a change of variable, the same as that for t_1, (3b) can be expressed as follows:

$$r_o^4 R(t) = n_o \sigma_a^2 \sum_{p=o}^{\gamma-1} (\gamma-p) e^{-j\omega_o t} \int \left[E\left\{ e^{-j\omega_o(q_o-q_p)} \right\} E\left\{ g(t_1,\tau)g(t_1+t-pT,\tau) \right\} \right.$$

$$\left. + E\left\{ q_o q_p e^{-j\omega_o(q_o-q_p)} \right\} E\left\{ \dot{g}(t_1,\tau)\dot{g}(t_1+t-pT,\tau) \right\} \right.$$

$$-E\left\{q_o e^{-j\omega_o(q_o-q_p)}\right\} E\left\{\overset{*}{g}(t_1,\tau)g(t_1+t-pT,\tau)\right\}$$

$$-E\left\{q_p e^{-j\omega_o(q_o-q_p)}\right\} E\left\{g(t_1,\tau)\overset{*}{g}(t_1+t-pT,\tau)\right\}\Big] . \qquad (7a)$$

The 3rd and 4th terms in (7a) cancel each other if the pulse envelope function $g(t_1)$ is even and if the conditions in (5a) and (5b) are true. Consequently (7a) is reduced to:

$$r_o^4 R(t) = n_o\sigma_a^2 \sum_{p=o}^{\gamma-1} (\gamma-p) e^{-j\omega_o t}\left[E\left\{e^{-j\omega_o(q_o-q_p)}\right\} r_1(t-pT)\right.$$

$$\left. + E\left\{q_o q_p e^{-j\omega_o(q_o-q_p)}\right\} r_2(t-pT)\right] , \qquad (7b)$$

where $r_1(t-pT) = \int p(\tau)d\tau \int g(t_1,\tau)g(t_1+t-pT,\tau)dt_1$, (8a)

and $r_2(t-pT) = \int p(\tau)d\tau \int \overset{*}{g}(t_1,\tau)\overset{*}{g}(t_1+t-pT,\tau)dt_1$. (8b)

3.6 THE ENERGY SPECTRAL-DENSITY FOR γ ECHO-SETS

The energy spectral density for γ echo-sets is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(t)\ \mathrm{Cos}\ \omega t dt, \qquad (9a)$$

where $R(t)$ is given by (7b).

Therefore:

$$\frac{r_o^4}{n_o\sigma_a^2} S(\omega) = \sum_{p=o}^{\gamma-1} (\gamma-p) \int_{-\infty}^{\infty} e^{-j\omega_o t}\ \mathrm{Cos}\omega t \left[E\left\{e^{-j\omega_o(q_o-q_p)}\right\} r_1(t-pT)\right.$$

$$\left. + E\left\{q_o q_p e^{-j\omega_o(q_o-q_p)}\right\} r_2(t_1-pT)\right] dt \qquad (9b)$$

By making the substitution t-pT = z; noting that $r_1(z)$ and $r_2(z)$ are even functions, and recalling that $e^{-j\omega_o pT} = 1$, (9b) becomes:

$$\frac{r_o^4}{n_o\sigma_a^2} S(\omega) = \sum_{p=o}^{\gamma-1} (\gamma-p)\ \mathrm{Cos}\omega pT \int_{-\infty}^{\infty} \mathrm{Cos}\omega_o z\ \ \mathrm{Cos}\omega z \left[E\left\{ e^{-j\omega_o(q_o-q_p)} \right\} r_1(z) \right.$$

$$\left. + E\left\{ q_o q_p e^{-j\omega_o(q_o-q_p)} \right\} r_2(z) \right] dz. \qquad (9c)$$

Now, separating off the terms in the summation for p=o and by introducing the functions $\Phi_1(\omega)$ and $\Phi_2(\omega)$, where

$$\Phi_1(\omega) = \int_{-\infty}^{\infty} \mathrm{Cos}\omega_o z\ \ \mathrm{Cos}\omega z r_1(z) dz \qquad (10a)$$

is the expected energy spectrum of a single echo-pulse envelope shifted from $\omega=o$ to $\omega=\omega_o$ with perfect stability assumed and

$$\Phi_2(\omega) = \int_{-\infty}^{\infty} \mathrm{Cos}\omega_o z\ \ \mathrm{Cos}\omega z r_2(z) dz \qquad (10b)$$

is the expected energy spectrum of the time derivative of a single echo-pulse envelope shifted from $\omega=o$ to $\omega=\omega_o$, also with perfect stability assumed, equation (9c) can be expressed as:

$$\frac{r_o^4}{n_o\sigma_a^2} S(\omega) = \gamma\left[\Phi_1(\omega)+\sigma_q^2\Phi_2(\omega)\right] + \sum_{p=1}^{\gamma-1} (\gamma-p)\mathrm{Cos}\omega pT \left[E\left\{ e^{-j\omega_o(q_o-q_p)} \right\} \Phi_1(\omega) \right.$$

$$\left. + E\left\{ q_o q_p e^{-j\omega_o(q_o-q_p)} \right\} \Phi_2(\omega) \right]. \qquad (11)$$

The autocorrelation functions $r_1(z)$ and $r_2(z)$ are defined in (8a) and 8b).

3.7 THE FUNCTIONAL RELATIONSHIP BETWEEN $\Phi_1(\omega)$ and $\Phi_2(\omega)$.

For a pulse waveform $u(t) = g(t)e^{-j\omega_o t}$ as originally assumed in (1), $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are functionally related.

If $G(\omega)$ is now the Fourier transform of the pulse envelope waveform $g(t)$, then:

$$G(\omega_1,\tau) = \int_{-\infty}^{\infty} g(t,\tau)e^{-j\omega t}dt \qquad (12a)$$

where τ is the pulse length. Also

$$g(t,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega_1,\tau)e^{j\omega_1 t}d\omega_1 \ . \qquad (12b)$$

$$\text{Thus } \frac{d}{dt}g(t,\tau)=\dot{g}(t,\tau)= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega_1,\tau)\omega_1 e^{j\omega_1 t}d\omega_1 \ . \qquad (12c)$$

The autocorrelation functions of $g(t,\tau)$ and $\dot{g}(t,\tau)$ can now be expressed in terms of their Fourier transforms $G(\omega_1,\tau)$ and $j\omega_1 G(\omega_1,\tau)$ respectively(Ref.4). From (8a) and (8b),

$$r_1(z) = \int p(\tau)d\tau \ \frac{1}{2\pi}\int_{-\infty}^{\infty} G^2(\omega_1,\tau) \ \ \text{Cos}\omega_1 z d\omega_1, \qquad (13a)$$

and

$$r_2(z) = \int p(\tau)d\tau \ \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega_1^2 G^2(\omega_1,\tau) \ \ \text{Cos}\omega_1 z d\omega_1. \qquad (13b)$$

Substituting (13a) and 13b) into (10a) and (10b) and expanding the cosine products into sums,where

$$\text{Cos}\omega_o a \ \ \text{Cos}\omega z \ \ \text{Cos}\omega_1 z =$$

$$\frac{1}{4}\Big\{\text{Cos } z(\overline{\omega_o+\omega} + \omega_1) + \text{Cos } z(\overline{\omega_o+\omega} - \omega_1)$$

$$+ \text{Cos } z([\omega_o-\omega]+\omega_1) + \text{Cos } z([\omega_o-\omega]-\omega_1)\Big\},$$

and since $2\pi \int_{-\infty}^{\infty} \text{Coszxdz} = \delta(x)$, then:

$$\Phi_1(\omega) = \frac{1}{4}\int p(\tau)d\tau \int_{-\infty}^{\infty} G^2(\omega_1,\tau)\Big\{\delta([\omega_o+\omega]+\omega_1)+\delta([\omega_o+\omega]-\omega_1)$$

$$+ \delta([\omega_o-\omega]+\omega_1)+\delta([\omega_o-\omega]-\omega_1)\Big\}\, d\omega_1$$

$$= \frac{1}{4}\int p(\tau)d\tau \Big[G^2(-[\omega_o+\omega],\tau)+G^2([\omega_o+\omega],\tau)$$

$$+ G^2(-[\omega_o-\omega],\tau) + G^2([\omega_o-\omega],\tau)\Big] .$$

$G^2(\omega_1,\tau)$ is an even function of ω_1, hence

$$\Phi_1(\omega) = \frac{1}{2}\int p(\tau)d\tau \Big[G^2(\omega_o+\omega,\tau) + G^2(\omega_o-\omega,\tau)\Big].$$

Similarly:

$$\Phi_2(\omega) = \frac{1}{2}\int p(\tau)d\tau \Big[(\omega_o+\omega)^2G^2(\omega_o+\omega,\tau) + (\omega_o-\omega)^2G^2(\omega_o-\omega,\tau)\Big].$$

If ω_o is much greater than the bandwidth of $G^2(\omega_o\pm\omega,\tau)$, (as it is in the experiment planned) then:

$$\Phi_1(\omega) = \int p(\tau)d\tau G^2(\omega_o-|\omega|,\tau), \qquad (14a)$$

and

$$\Phi_2(\omega) = \int p(\tau)d\tau(\omega_o-|\omega|)^2G^2(\omega_o-|\omega|,\tau). \qquad (14b)$$

Consequently

$$\Phi_2(\omega) = (\omega_o-|\omega|)^2\Phi_1(\omega). \qquad (15)$$

The spectral density from (11) can now be expressed as:

$$S(\omega)=\frac{n_o\sigma_a^2}{r_o^4}\,\Phi_1(\omega)\left\{\gamma(1+(\omega_o-|\omega|)^2\sigma_q^2)+\sum_{p=1}^{\gamma-1}(\gamma-p)\cos\,\omega pT\;E\left\{e^{-j\omega_o(q_o-q_p)}\right\}\right.$$

$$\left.+(\omega_o-|\omega|)^2E\left\{q_oq_pe^{-j\omega_o(q_o-q_p)}\right\}\right\}. \qquad (16)$$

3.8 THE SPECTRAL-DENSITY FOR INDEPENDENT ECHO-SETS.

A considerable simplification ensues if the jitter variable q is independent of the echo-set difference number p. Then from (5a),

$$p(q_o,q_p)=p(q_o)p(q_p).$$

$$\text{Then } E\left\{e^{-j\omega_o(q_o-q_p)}\right\}=M_q(j\omega_o)M_q^*(j\omega_o)=|M_q(j\omega_o)|^2,$$

$$\text{and } E\left\{q_oq_pe^{-j\omega_o(q_o-q_p)}\right\}=\frac{d}{d\omega_o}M_q(j\omega_o)\frac{d}{d\omega_o}M_q^*(j\omega_o)=\left|\frac{d}{d\omega_o}M_q(j\omega_o)\right|^2,$$

where $M_q(j\omega_o)$ is the characteristic function of the jitter-variable q evaluated at ω_o and defined as $M_q(j\omega_o)=E\left\{e^{-j\omega_oq}\right\}$.

Using these results, the spectral density of (16) is written as:

$$\frac{r_o^4}{n_o\sigma_a^2}S(\omega)=\gamma\left[1+\sigma_q^2(\omega_o-|\omega|)^2\right]\Phi_1(\omega)$$

$$+\Phi_1(\omega)\sum_{p=1}^{\gamma-1}(\gamma-p)\cos\omega pT\left\{|M_q(j\omega_o)|^2+\left|\frac{d}{d\omega_o}M_q(j\omega_o)\right|^2(\omega_o-|\omega|)^2\right\}. \quad (1$$

Using the identity $\sum_{p=1}^{\gamma-1}(\gamma-p)\cos\,\omega pT=\dfrac{\sin^2\frac{\gamma\omega T}{2}}{\sin^2\frac{\omega T}{2}}-\gamma,$

(17a) becomes:

$$\frac{r_o^4}{n_o\sigma_a^2} S(\omega) = \Phi_1(\omega) \left\{ \gamma \left[(1-|M_q(j\omega_o)|^2) + (\omega_o - |\omega|)^2 (\sigma_q^2 - |\frac{d}{d\omega_o} M_q(j\omega_o)|^2) \right] \right.$$

$$\left. + \left[|M_q(j\omega_o)|^2 + (\omega_o - |\omega|)^2 |\frac{d}{d\omega_o} M_q(j\omega_o)|^2 \right] \frac{\text{Sin}^2 \frac{\gamma\omega T}{2}}{\text{Sin}^2 \frac{\omega T}{2}} \right\} . \qquad (17b)$$

In this expression, the first term represents the incoherent contribution to the spectral density and includes a term to indicate the extent of broadening of the incoherent component. The second term is a "comb" spectrum and will exist if the medium has some stability.

For an unstable medium, $M_q(j\omega_o) \to o$ and $\frac{d}{d\omega_o} M_q(j\omega_o) \to o$,

so that $S(\omega) \to S_u(\omega) = \frac{n_o\sigma_a^2}{r_o^4} \gamma \left[1 + (\omega_o - |\omega|)^2 \sigma_q^2 \right] \Phi_1(\omega)$ (18)

Also, for stability, $M_q(j\omega_o) \to 1$, $\frac{d}{d\omega_o} M_q(j\omega_o) \to o$, and $\sigma_q \to o$

so that

$$S(\omega) \to S_c(\omega) = \frac{n_o\sigma_a^2}{r_o^4} \frac{\text{Sin}^2 \frac{\gamma\omega T}{2}}{\text{Sin}^2 \frac{\omega T}{2}} \Phi_1(\omega). \qquad (19)$$

3.9 GAUSSIAN VARIABLES

Returning to the more general expression of equation (16), and if the jitter variables are assumed to be Gaussian, then the joint probability density in (5a) can be expressed as (Ref. 5):

$$p(q_o, q_p) = (2\pi)^{-1} \sigma_q^{-2} (1-\rho^2)^{-\frac{1}{2}} \exp \left[- \frac{q_o^2 - 2\rho q_o q_p + q_p^2}{2\sigma_q^2 (1-\rho^2)} \right] . \qquad (20)$$

- - - - - - - - - - -

In (20), q_o and q_p have zero mean values and identical variances σ_q. The correlation coefficient of q_o and q_p is ρ and must be a function of the echo-set number p so that $\rho = \rho(p)$.

Applying (20) to the expectations in (16) results in:

$$E\left\{e^{-j\omega_o(q_o-q_p)}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(q_o,q_p)e^{-j\omega_o(q_o-q_p)}\,dq_o\,dq_p,$$

and

$$E\left\{q_oq_pe^{-j\omega_o(q_o-q_p)}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_oq_pp(q_o,q_p)e^{-j\omega_o(q_o-q_p)}\,dq_o\,dq_p.$$

After some reduction, the energy spectral density of (16) becomes:

$$\frac{r_o^4}{n_o\sigma_a^2}S(\omega) = \Phi_1(\omega)\left\{\gamma\left(1+(\omega_o-|\omega|)^2\sigma_q^2\right)\right.$$

$$\left. +e^{-\omega_o^2\sigma_q^2}\sum_{p=1}^{\gamma-1}(\gamma-p)e^{\omega_o^2\sigma_q^2\rho}\cos\omega pT\left[1+(\omega_o+|\omega|)^2\sigma_q^2(\rho+\sigma_q^2\omega_o^2(1-3\rho)(1-\rho))\right]\right\} \tag{21a}$$

where $\rho = \rho(p)$.

For independent echo-sets and Gaussian variables, (21a) reduces to

$$\frac{r_o^4}{n_o\sigma_a^2}S(\omega) = \Phi_1(\omega)\left\{\gamma\left[1+(\omega_o-|\omega|)^2\sigma_q^2\right] + e^{-\omega_o^2\sigma_q^2}\left[\frac{\sin^2\frac{\gamma\omega T}{2}}{\sin^2\frac{\omega T}{2}} - \gamma\right](1+(\omega_o-|\omega|)^2\sigma_q^4\omega_o^2)\right\}. \tag{21b}$$

A sample calculation of equation 21b is given in Appendix A and the results are plotted in Figure 8 .

It is difficult to see at a glance what the effect of the

dependence of q_p on q_o implies in equation (21a) where correlation coefficient ρ is not zero. Intuitively, if ρ is not zero, then one might expect a small transfer of incoherent energy to coherent energy equivalent to a reduction in the jitter variance σ_q. This effect can be illustrated by choosing a function $\rho(p)$ which is analytically tractable in (21a). Therefore, let $\rho = \{1+\text{Cos}(\pi p/(\gamma-1))\}d/2$ where $d<<1$. For $\gamma-1 \geq p \geq 1$, ρ monotonically decreases.

Letting $\omega_q^2 \sigma_q^2 = \xi^2$ where ξ is the phase-angle jitter variance and also neglecting the insignificant higher order spectral spreading corrections, then (21a) becomes:

$$\frac{r_o^4}{n_o \sigma_a^2} S(\omega) = S_o(\omega) = \Phi_1(\omega) \left\{ \gamma + e^{-\xi^2} \sum_{p=1}^{\gamma-1} (\gamma-p) e^{\frac{\xi^2 d}{2}(1+\text{Cos}[p\pi/(\gamma-1)])} \text{Cos}\ \omega pT \right\}$$

$$\approx \Phi_1(\omega) \left\{ \gamma + e^{-\xi^2} \sum_{p=1}^{\gamma-1} (\gamma-p) \left[1+\frac{\xi^2 d}{2}(1+\text{Cos}\left[p\pi/(\gamma-1)\right]) \right] \text{Cos}\omega pT \right\}. \quad (22a)$$

Using the identity $\sum_{p=1}^{\gamma-1} (\gamma-p)\text{Cos}\omega pT = \left(\frac{\text{Sin}^2 \frac{\gamma\omega T}{2}}{\text{Sin}^2 \frac{\omega T}{2}} - \gamma \right)$,

(22a) becomes:

$$S_o(\omega) \approx \Phi_1(\omega) \left\{ \gamma(1-e^{-\xi^2}\left[1+\xi^2 d\right]) + e^{-\xi^2} \left(\left[1+\frac{\xi^2 d}{2}\right] \frac{\text{Sin}^2 \frac{\gamma\omega T}{2}}{\text{Sin}^2 \frac{\omega T}{2}} + \frac{\xi^2 d}{2} \frac{\text{Cos}^2 \frac{\gamma\omega T}{2}}{\text{Cos}^2 \frac{\omega T}{2}} \right) \right\}. \quad (22b)$$

In this illustration, the effect of dependence is to transfer incoherent energy to the existing comb spectrum and also into a quadrature comb spectrum.

4.0 A CRITERION FOR STABILITY

A criterion for acoustic stability of the medium is now defined as the ratio of the coherent energy to the total sound energy in a sequence of range-gated reverberation returns. The criterion as defined will be range-dependent and is given the cymbol C_r. It has been shown in the section 3.9, by means of a particular example, that if a jitter-variable dependence exists between echo-sets, then for the same jitter variance the medium will exhibit a higher stability. In the interests of simplicity it is therefore useful to think of C_r as an experimentally measured factor which can be equated to an analytical expression derived under the assumption of echo-set

independence. If an echo-set dependence exists in fact, then the experimentally determined C_r should also exhibit a dependence on the pulse repetition period T.

For echo-set independence and for narrow-band transmitted pulses, it will be shown that $C_r \approx |M_q(j\omega_o)|^2$, where $M_q(j\omega_o)$ is the characteristic function of the jitter-variable q evaluated at ω_o (see section 3.8). From 17b, the total energy is given by:

$$\frac{r_o^4}{n_o \sigma_a^2} \int_{-\infty}^{\infty} S(\omega) d\omega = \varepsilon_u + \varepsilon_c ,$$

where

$$\varepsilon_u = \gamma(1-|M_q(j\omega_o)|^2) \int_{-\infty}^{\infty} \Phi_1(\omega) d\omega$$

$$+ \gamma(\sigma_q^2 - |\frac{d}{d\omega_o} M_q(j\omega_o)|^2) \int_{-\infty}^{\infty} (\omega_o - |\omega|)^2 \Phi_1(\omega) d\omega \qquad (23a)$$

represents the incoherent energy, and

$$\varepsilon_c = |M_q(j\omega_o)|^2 \int_{-\infty}^{\infty} \Phi_1(\omega) \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}} d\omega$$

$$+ |\frac{d}{d\omega_o} M_q(j\omega_o)|^2 \int_{-\infty}^{\infty} (\omega_o - |\omega|)^2 \Phi_1(\omega) \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}} d\omega \qquad (23b)$$

represents the coherent energy.

In (23b), the functions $\Phi_1(\omega)$ and $(\omega_o - |\omega|)^2 \Phi_1(\omega)$ vary slowly compared to $\mathrm{Sin}^2(\gamma\omega T/2)/\mathrm{Sin}^2(\omega T/2)$ as can be seen in Figure 8. Therefore, the integrals can be re-expressed as summations of integrals of the "teeth of the comb" over a limited range of ω as follows:

$$\varepsilon_c = |M_q(j\omega_o)|^2 \sum_{\ell=-\infty}^{\infty} \Phi_1\left(\frac{2\pi\ell}{T}\right) \int_{-\delta\omega}^{\delta\omega} \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}}\, d\omega$$

$$+ \left|\frac{d}{d\omega_o} M_q(j\omega_o)\right|^2 \sum_{\ell=-\infty}^{\infty} \left(\omega_o - \left|\frac{2\pi\ell}{T}\right|\right)^2 \Phi_1\left(\frac{2\pi\ell}{T}\right) \int_{-\delta\omega}^{\delta\omega} \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}}\, d\omega \qquad (23c)$$

where $\frac{2\pi}{\gamma T} \leq \delta\omega << \frac{2\pi}{T}$.

Over the range $-\delta\omega \leq \omega \leq \delta\omega$, $\mathrm{Sin}^2 \frac{\omega T}{2} \approx \left(\frac{\omega T}{2}\right)^2$

so that:

$$\int_{-\delta\omega}^{\delta\omega} \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}}\, d\omega \approx \int_{-\infty}^{\infty} \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\left(\frac{\omega T}{2}\right)^2}\, d\omega = \frac{2\pi\gamma}{T}$$

Recognizing that

$$\ell = \sum_{\ell=-\infty}^{\infty} \Phi_1\left(\frac{2\pi\ell}{T}\right)\frac{2\pi}{T} \approx \int_{-\infty}^{\infty} \Phi_1(\omega)\, d\omega$$

then (23c) can be written as:

$$\varepsilon_c \approx \gamma |M_q(j\omega_o)|^2 \int_{-\infty}^{\infty} \Phi_1(\omega)\, d\omega + \gamma \left|\frac{d}{d\omega_o} M_q(j\omega_o)\right|^2 \int_{-\infty}^{\infty} (\omega_o - |\omega|)^2\, \Phi_1(\omega)\, d\omega. \qquad (23d)$$

The stability criterion is defined as $\varepsilon_c/(\varepsilon_u+\varepsilon_c)$; consequently from (23a) and (23d):

$$C_r = \frac{|M_q(j\omega_o)|^2 \int_{-\infty}^{\infty} \Phi_1(\omega)\, d\omega + \left|\frac{d}{d\omega_o} M_q(j\omega_o)\right|^2 \int_{-\infty}^{\infty} (\omega_o - |\omega|)^2 \Phi_1(\omega)\, d\omega}{\int_{-\infty}^{\infty} \Phi_1(\omega)\, d\omega + \sigma_q^2 \int_{-\infty}^{\infty} (\omega_o - |\omega|)^2 \Phi_1(\omega)\, d\omega} .$$

For $\Phi_1(\omega)$ narrow-band and centred on ω_o, the above expression simplifies to:

$$C_r \approx |M_q(j\omega_o)|^2 \tag{24a}$$

If the time jitter variable is Gaussian,

$$C_r \approx \exp\left[-\omega_o^2 \sigma_q^2\right] \tag{24b}$$

In the latter event, if the jitter variance $\sigma_q = o$, then $C_r = 1$ which is the condition for complete stability.

4.1 EXPERIMENTAL MEASUREMENT OF C_r BY SAMPLING.

From the sampling theorem, the mth echo-set waveform is:

$$k_m(t_1 - mT) = \sum_{\alpha=o}^{\alpha_K - 1} g_m(\alpha\Gamma - mT) \frac{\mathrm{Sin}\omega_2(t_1 - \alpha\Gamma - mT)}{\omega_2(t_1 - \alpha\Gamma - mT)}, \tag{25}$$

where α_K equal-interval samples are required to fully define the mth echo-set, the sampling period $\Gamma = \pi/\omega_2$, where ω_2 is at least 2π times the upper frequency limit of $k_m(t_1 - mT)$ and the time mT is an integral multiple of the sampling period Γ. The sample amplitude is $g_m(\alpha\Gamma - mT)$ and can be rewritten as:

$$g_m(\alpha\Gamma - mT) = \bar{g}(\alpha\Gamma) + \lambda(\alpha\Gamma - mT), \tag{26}$$

where

$$\bar{g}(\alpha\Gamma) = \frac{1}{\gamma} \sum_{m=0}^{\gamma - 1} g_m(\alpha\Gamma - mT). \tag{27}$$

In (26), $\bar{g}(\alpha\Gamma)$ is defined as the arithmetic average of all the αth samples over all γ echo-sets and can be thought of as the "coherent" αth sample. Also, $\lambda(\alpha\Gamma - mT)$ is now a random variable associated with the incoherent energy in the γ echo-sets.

Specifically, the following expectations are understood:

$$E\left\{\lambda(\alpha\Gamma - mT)\right\} = o, \tag{28a}$$

and

$$E\left\{\lambda^2(\alpha\Gamma - mT)\right\} = \sigma_\lambda^2 . \tag{28b}$$

From (25) and (26) the γ echo-sets (see Figure 6) can now be represented as:

$$\sum_{m=o}^{\gamma-1} k_m(t_1-mT) = \sum_{m=o}^{\gamma-1}\sum_{\alpha=o}^{\alpha_K-1} (\bar{g}(\alpha\Gamma) + \lambda(\alpha\Gamma - mT)) \frac{\text{Sin } \omega_2(t_1-\alpha\Gamma - mT)}{\omega_2(t_1-\alpha\Gamma - mT)} . \quad (29)$$

The total energy in the waveform of (29) is proportional to the time integral of the statistical expectation of the amplitude squared, i.e.;

$$\varepsilon = \int_{-\infty}^{\infty} E\left\{ \left(\sum_{m=0}^{\gamma-1} k_m(t_1-mT) \right)^2 \right\} dt_1 . \quad (30a)$$

Since $\lambda(\alpha\Gamma - mT)$ is a random variable with a zero mean, then from 28a, the cross-product terms vanish when the rhs of (29) is inserted in the expression (30a) with the result that:

$$\varepsilon = \sum_{m=o}^{\gamma-1}\sum_{\alpha=o}^{\alpha_K-1}\sum_{n=o}^{\gamma-1}\sum_{\beta=o}^{\alpha_K-1} \left[\bar{g}(\alpha\Gamma)\,\bar{g}(\beta\Gamma) + E\left\{ \lambda(\alpha\Gamma - mT)\lambda(\beta\Gamma - nT) \right\} \right] \times$$

$$\int_{-\infty}^{\infty} \frac{\text{Sin } \omega_2(t_1-\alpha\Gamma - mT)}{\omega_2(t_1-\alpha\Gamma - mT)} \cdot \frac{\text{Sin } \omega_2(t_1-\beta\Gamma - nT)}{\omega_2(t_1-\beta\Gamma - nT)} dt_1 . \quad (30b)$$

The latter integral is zero except for the condition $\alpha\Gamma+mT=\beta\Gamma+nT$; then its value is π/ω_2 . Also since the time interval T is much larger than $\alpha_K\Gamma$, $\alpha\Gamma = \beta\Gamma$ and $mT = nT$, consequently (30b) becomes;

$$\varepsilon = \frac{\pi}{\omega_2} \sum_{m=o}^{\gamma-1}\sum_{\alpha=o}^{\alpha_K-1} \left[\bar{g}^2(\alpha\Gamma) + E\left\{ \lambda^2(\alpha\Gamma - mT) \right\} \right] \quad (30c)$$

$$= \frac{\gamma\pi}{\omega_2} \left[\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^2(\alpha\Gamma) + \alpha_K\sigma_\lambda^2 \right] . \quad (30d)$$

The stability criterion C_r, as defined in section4.0, can now be expressed in terms of the sample amplitude using (30d);

$$C_r = \frac{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma)}{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma) + \alpha_K\sigma_\lambda^2} \cdot \qquad (31a)$$

Alternatively C_r can be expressed in the form:

$$C_r = \frac{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma)}{\sum_{\alpha=o}^{\alpha_K-1} E\left\{g_m^{\,2}(\alpha\Gamma - mT)\right\}} \cdot \qquad (31b)$$

where $\bar{g}(\alpha\Gamma)$ is defined in (27), with α_K being the number of samples required to describe an echo-set according to (25), and σ_λ is the variance of the random sample difference $\lambda(\alpha\Gamma - mT)$ defined in (26).

By equating (24a) and (31a, 31b), the square of the absolute value of the characteristic function of the time jitter variable can be computed:

$$|M_q(j\omega_o)|^2 \approx \frac{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma)}{\sum_{\alpha=0}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma) + \alpha_K\sigma_\lambda^2} = \frac{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{\,2}(\alpha\Gamma)}{\sum_{\alpha=o}^{\alpha_K-1} E\left\{g_m^{\,2}(\alpha\Gamma - mT)\right\}} \cdot \qquad (32)$$

As discussed in section 4.0, the value of $|M_q(j\omega_o)|^2$ may depend on the pulse repetition period T if a statistical dependence between echo-sets exists. Computation of the time-jitter variance σ_q cannot be directly done from (32) unless the jitter probability density function is assumed or measured by another method. If q is a Gaussian variable, as in section 3.9, then $|M_q(j\omega_o)|^2 = \exp\left[-\omega_o^2\sigma_q^2\right]$.

On the other hand, an estimate of the probability density function p(q) could be obtained by a measure of the time-jitter along a single transmission path.

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5.0 SUMMARY

(1) A preliminary test of sound-path stability made in M'Clure Strait in 1966 showed that the phase stability at 4 kHz in direct sound-paths to a distance of ½ mile was beyond the capability of the instrumentation used to measure it.

(2) A theoretical method of assessing the acoustic stability of under-ice sound transmissions has been developed. If a precisely controlled set of pulse-sounds are projected into the under-ice medium being tested, then the information to be analyzed consists of a set of reverberation signals. If all the reverberation signals are precisely identical, the medium is stable, at least to the ranges where the reverberations are well above system or ambient noise. If some instability exists it is likely to be range dependent. Hence the theory has been framed in terms of a set of range-gated reverberation signal or "echo-sets" as shown in Figure 6. It has been shown that the reverberation samples or echo-sets will consist of a coherent, or non-noise, component plus an incoherent, or noise, component. The proportioning of reverberation energy between the two components will depend on the statistical properties of the time or phase-angle jitter over a single sound-path to the same range.

In a particular case and for a Gaussian time-jitter variable (21b), the energy spectral density for γ echo-sets is expressed as:

$$S_o(\omega) = \Phi_1(\omega) \cdot \gamma \left[1 + (\omega_o - |\omega|)^2 \sigma_q^2\right]$$

$$+ \exp\left[-\omega_o^2 \sigma_q^2\right] \cdot \frac{\mathrm{Sin}^2 \frac{\gamma\omega T}{2}}{\mathrm{Sin}^2 \frac{\omega T}{2}} - \gamma \left(1 + (\omega_o - |\omega|)^2 \sigma_q^4 \omega_o^2\right) .$$

Here the expected spectrum of a single echo is $\Phi_1(\omega)$ which is multiplied by two separate terms. The first term is the "incoherent" contribution which is proportional to the number of echo-sets, γ, and contains a spectral spreading factor $(\omega_o - |\omega|)^2 \sigma_q^2$, where σ_q is the variance of the time-jitter over a single path. This spreading factor also reappears in a slightly different form in the second term or "coherent" contribution.

The function $\mathrm{Sin}^2 \frac{\gamma\omega T}{2} / \mathrm{Sin}^2 \frac{\omega T}{2}$ in the coherent term forms a set of extremely sharp peaks or "combs" in the spectrum for circular frequency intervals of $2\pi/T$. It can be seen that the proportion of coherent energy depends on the factor $\exp\left[-\omega_o^2 \sigma_q^2\right]$ and hence on the time-jitter variance σ_q.

(3) By experimentally sampling a particular set of range-gated reverberations a stability criterion C_r can be computed. This criterion is simply related to the statistical properties of the single path time-jitter as well as to the analytical form of the energy-spectrum of the set. Specifically, the criterion C_r is specified in terms of the echo-set sample amplitudes (26): $g_m(\alpha\Gamma - mT)$ where m is the echo-set number and α is the sample number in the mth echo-set, so that (31b):

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$$C_r = \frac{\sum_{\alpha=o}^{\alpha_K-1} \bar{g}^{-2}(\alpha\Gamma)}{\sum_{\alpha=o}^{\alpha_K-1} E\left\{g^2 m(\alpha\Gamma - mT)\right\}}, \quad \text{for } \bar{g}^{-2}(\alpha\Gamma) = \frac{1}{\gamma}\sum_{m=o}^{\gamma-1} g_m(\alpha\Gamma - mT),$$

and $m=o,1,2 - - -(\gamma-1)$, $\alpha=o,1,2 - - - \alpha_K-1$.

The criterion can also be related to the time-jitter variance σ_q. In the particular case where the time-jitter variable in Gaussian(24b)

$$C_r = \exp\left[-\omega_o^2\sigma_q^2\right]$$

(4) Although statistical stationarity may exist over the time interval required to project γ pulses, it is anticipated that the experimental analysis of γ echo-sets at different times of day and on different days may reveal significant differences in the range-dependent stability criterion C_r.

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7.0 LIST OF SYMBOLS

a_α	echo or wavelet amplitude from the αth scatterer.
α,β	integers; scatterer number; sampling number.
C_r	stability criterion.
c	speed of sound.
d	constant.
$E\{\ \}$	statistical expectation.
ξ	phase-jitter variance
ε	total energy in γ echo-sets.
ε_c	coherent energy in γ echo-sets.
ε_u	incoherent energy in γ echo-sets.
$\Phi_1(\omega)$	expected energy-spectrum of a single echo-pulse envelope, frequency shifted to ω_o
$\Phi_2(\omega)$	expected energy-spectrum of the time-derivative of a single echo-pulse envelope, frequency shifted to ω_o
$g(t_1)$	pulse-envelope waveform.
$G(\omega_1)$	Fourier transform of $g(t_1)$.
$g_m(\alpha\Gamma-mT)$	sample amplitude.
$\overline{g}(\alpha\Gamma)$	average αth sample amplitude over γ echo-sets.
γ	total number of echo-sets
K	gated-interval time.
$k_m(t_1)$	waveform of the mth echo-set.
k	constant; bandwidth parameter.
$\lambda(\alpha\Gamma-mT)$	random sample amplitude difference.
Γ	sampling period.
m,n	integers; gated interval or echo-set numbers.

$M_q(j\omega)$	characteristic function of q.
n_o	average number of echoes in an echo-set.
p	summation integer.
p(q),p(a)	probability density functions.
q	time-jitter variable.
R(t)	autocorrelation function of γ echo-sets.
ΔR	width of surface annular area.
$r_1(t)$	autocorrelation function of the pulse envelope.
$r_2(t)$	autocorrelation function of the time-derivative of the pulse envelope.
r_o	slant range
ρ	correlation coefficient
$S(\omega), S_o(\omega)$	energy-density spectrum of γ echo-sets.
$\sigma_q, \sigma_a, \sigma_\lambda$	variances
T	pulse repetition period.
t, t_1, t_α	time variables.
τ_o	transmitted pulse length.
τ_α	echo-pulse length.
$u(t_1)$	pressure-pulse waveform.
ω, ω_o	circular frequency.
$\Delta\omega$	circular frequency difference.
z	time variable.

8.0 REFERENCES

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Figure 2 Arrangement of experimental apparatus in April and May, 1966.

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10.0 APPENDIX

Sample Calculation of S(ω) from Equation (21b)

Repeating equation (21b):

$$\frac{r_o^4}{n_o \sigma_a^2} S(\omega) = S_o(\omega) = \Phi_1(\omega) \left\{ \gamma \left[1+(\omega_o-|\omega|)^2\sigma_q^2 \right. \right.$$

$$\left. \left. + e^{-\omega_o^2\sigma_q^2} \left[\frac{\text{Sin}^2 \frac{\gamma\omega T}{2}}{\text{Sin}^2 \frac{\omega T}{2}} - \gamma \right] 1+(\omega_o-|\omega|)^2\sigma_q^4\omega_o^2 \right. \right. .$$

In (1A), $S_o(\omega)$ includes the scale factor $r_o^4/(n_o\sigma_a^2)$ where r_o is the mean range to the region which includes n_o scatterers each with an echo amplitude variance σ_a.

Assuming that the output pulse u(t) is formed by gating a sinusoidal wave-form of circular frequency ω_o and that the whole sending and receiving bandwidth can be represented by a bell-shaped filter response, then the resulting envelope function $g(t,\tau)$ will have a Fourier transform

$$G(\omega,\tau) = \frac{2\text{Sin}(\frac{\omega\tau}{2})}{\omega} e^{-\frac{\omega^2k^2}{2}} \quad - - - - - \quad (2A)$$

where the function $\exp(-\omega^2k^2/2)$ represents the filter response of the system.

From (14a), the function $\Phi_1(\omega)$ is:

$$\Phi_1(\omega) = \int p(\tau) \frac{4\text{Sin}^2 \left[(\omega_o-|\omega|)\tau/2\right]}{(\omega_o-|\omega|)^2} \exp\left[-(\omega_o-|\omega|)^2k^2\right] d\tau. - - - - \quad (3A)$$

It can be seen from Figure 7 that the echo-waveforms near the edge of the gated interval are truncated. This effect can be accounted for by letting

$$p(\tau) = \frac{1}{K} + \frac{K-\tau_o}{K} \delta(\tau-\tau_o), \; 0 \leq \tau \leq \tau_o, \; - - - - \quad (4A)$$

where τ_o is the transmitted pulse length.

Substituting (4a) in (3A),:

$$\Phi_1(\omega) = 4 \exp\left[-(\omega_o-|\omega|)^2k^2\right] \ (\omega_o-|\omega|)^{-2} \times$$

$$\left\{\frac{1}{K}\int_o^{\tau_o} \mathrm{Sin}^2\left[(\omega_o-|\omega|)\tau/2\right] d\tau \right.$$

$$\left. + \frac{K-\tau_o}{K} \mathrm{Sin}^2\left[(\omega_o-|\omega|)\tau_o/2\right]\right\},$$

so that

$$\Phi_1(\omega) = 4(\omega_o-|\omega|)^{-2}\exp\left[-(\omega_o-|\omega|)^2k^2\right] \times$$

$$\left\{\frac{\tau_o}{2k}\left[1- \frac{\mathrm{Sin}\left[(\omega_o-|\omega|)\tau_o\right]}{(\omega_o-|\omega|)\tau_o}\right]+\frac{K-\tau_o}{K} \mathrm{Sin}^2\left[(\omega_o-|\omega|)\tau_o/2\right]\right\}. \qquad (5A)$$

The quantity $\omega_o\sigma_q$ is the variance of the phase jitter, in radians. Letting $\omega_o\sigma_q=\xi$, (1A) can be re-expressed as:

$$S_o(\omega)= \Phi_1(\omega)\left\{\gamma\left[1 + \frac{\omega_o-|\omega|}{\omega_o}\right)^2 \xi^2\right]$$

$$+ e^{-\xi^2}\left[\frac{\mathrm{Sin}^2\frac{\gamma\omega T}{2}}{\mathrm{Sin}^2\frac{\omega T}{2}} -\gamma\right]\left[1 +\left(\frac{\omega_o-|\omega|}{\omega_o}\right)^2 \xi^4\right]\right\}. \qquad (6A)$$

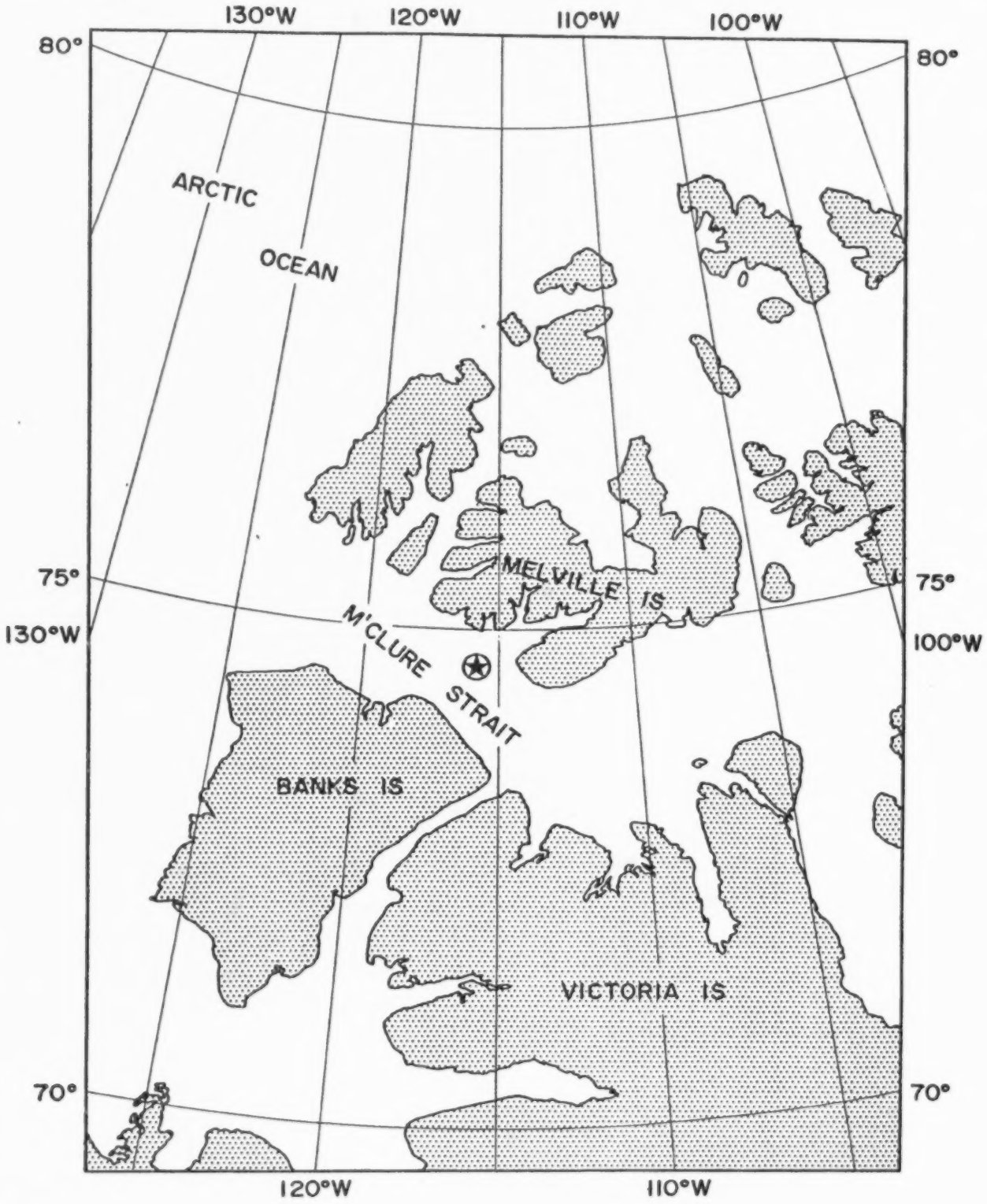
Letting $|\omega_o-|\omega|| = \Delta\omega$ and noting that $\mathrm{Sin}^2(\gamma\omega T/2) = \mathrm{Sin}^2\left[\gamma(\omega_o-|\omega|)T/2\right]$,

from (5A) and (6A) we obtain:

$$S_o(\omega) = 4\Delta\omega^{-2}\exp\left[-\Delta\omega^2 k^2\right]\left\{\frac{\tau_o}{2K}\left[1-\frac{\mathrm{Sin}\Delta\omega\tau_o}{\Delta\omega\tau_o}\right]+\frac{K-\tau_o}{K}\mathrm{Sin}^2\left[\Delta\omega\tau_o/2\right]\right\}\times$$

$$\left(\gamma\left[1+\left(\frac{\Delta\omega}{\omega_o}\right)^2\xi^2\right]+\exp\left[-\xi^2\right]\left[\frac{\mathrm{Sin}^2(\gamma\Delta\omega T/2)}{\mathrm{Sin}^2(\Delta\omega T/2)}-\gamma\left[1+\left(\frac{\Delta\omega}{\omega_o}\right)^2\xi^4\right]\right]\right. \quad (7A$$

Figure 8 is a plot of (7A) with the following parameters assumed: $\xi = \pi/2$ radians, $\gamma = 39$, $\omega_o/2\pi = 1\times10^3$ Hertz, $T = 5.0$ sec, $\tau_o = 0.1$ sec, $K/\tau_o = 5$ and $(k/\tau_o)^2 = .0175$. This value for $(k/\tau_o)^2$ is determined by assuming that $\exp\left[-\Delta\omega^2k^2\right] = \frac{1}{2}$ for $\Delta\omega = 2\pi/\tau_o$.



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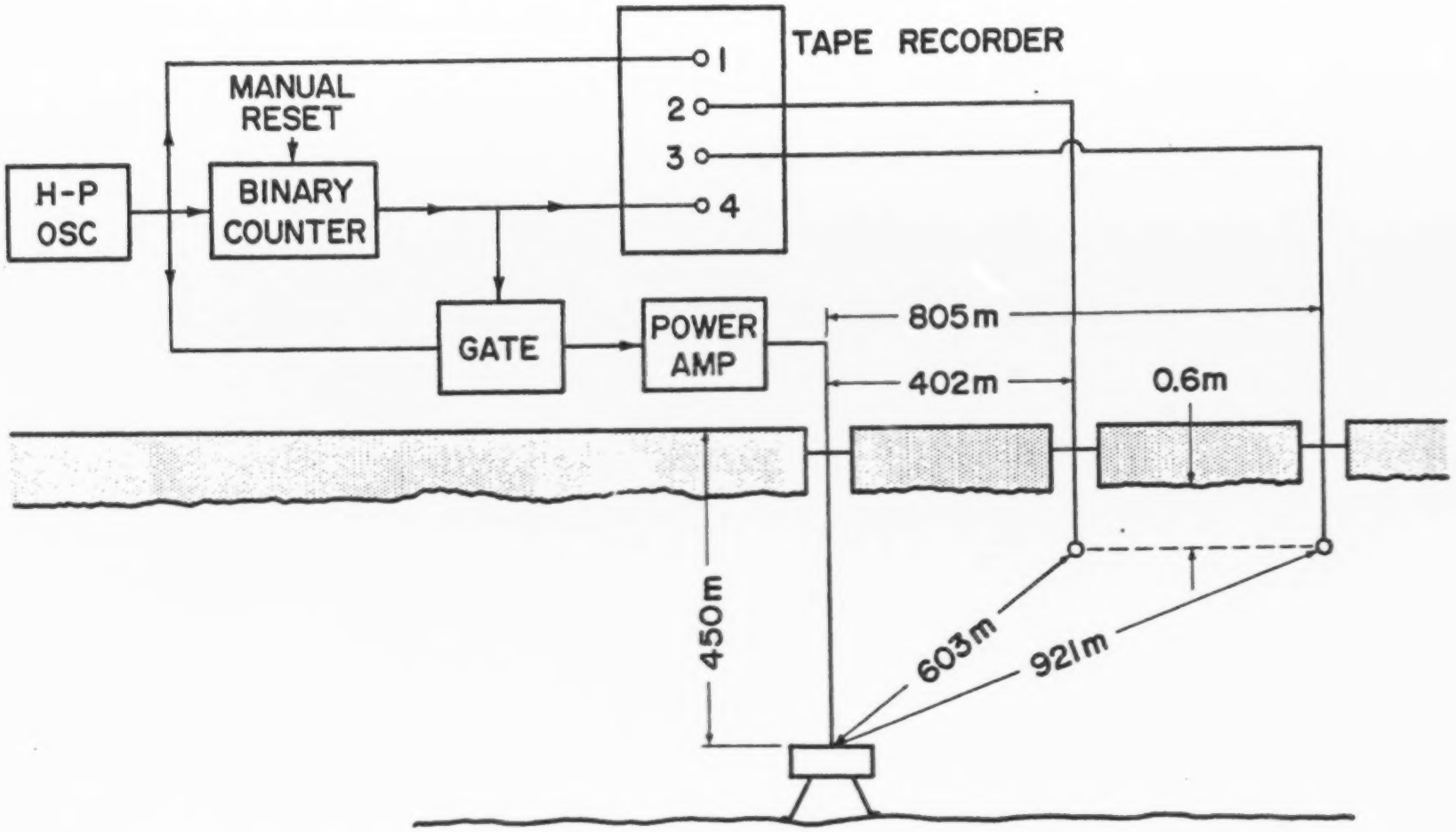

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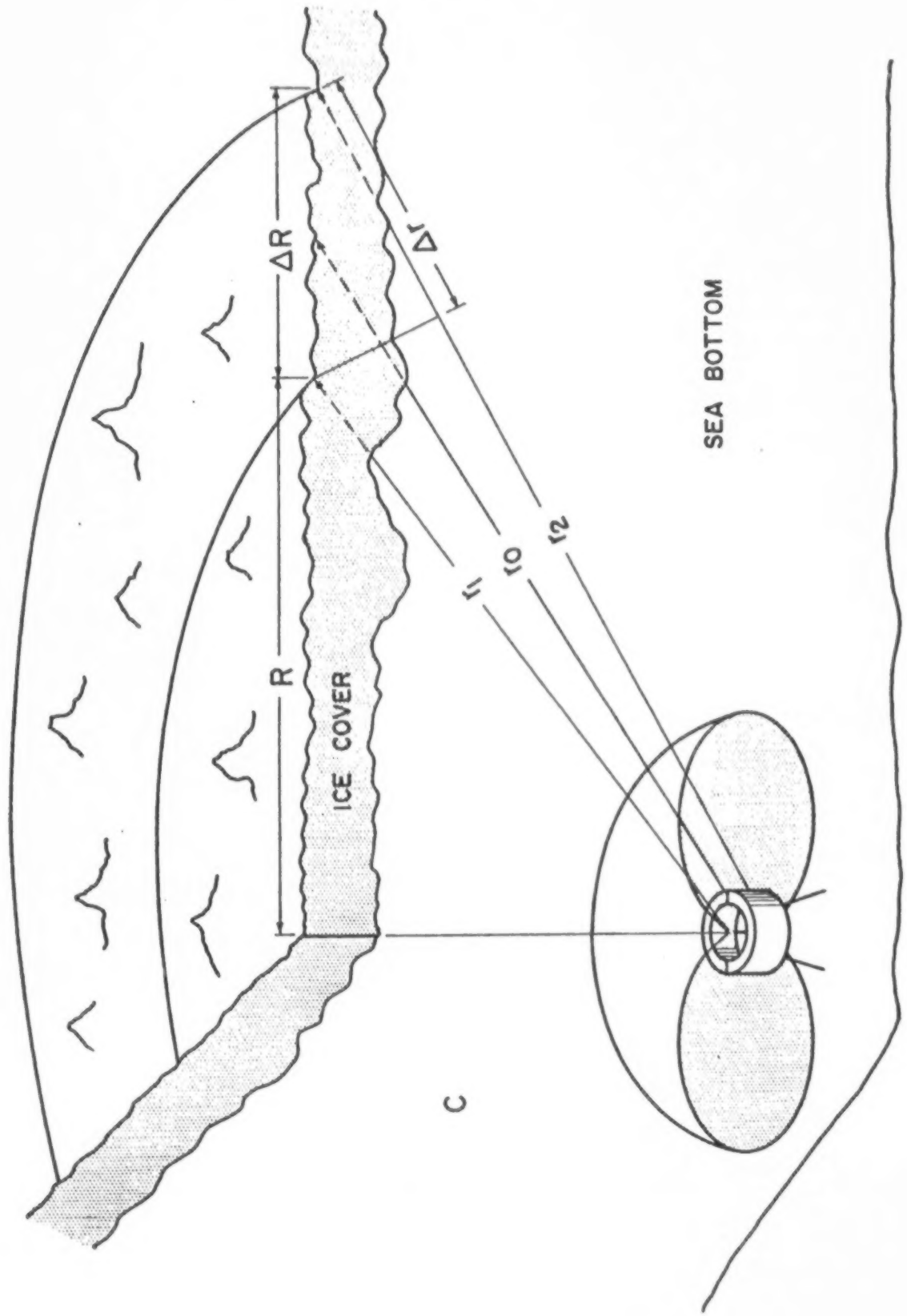

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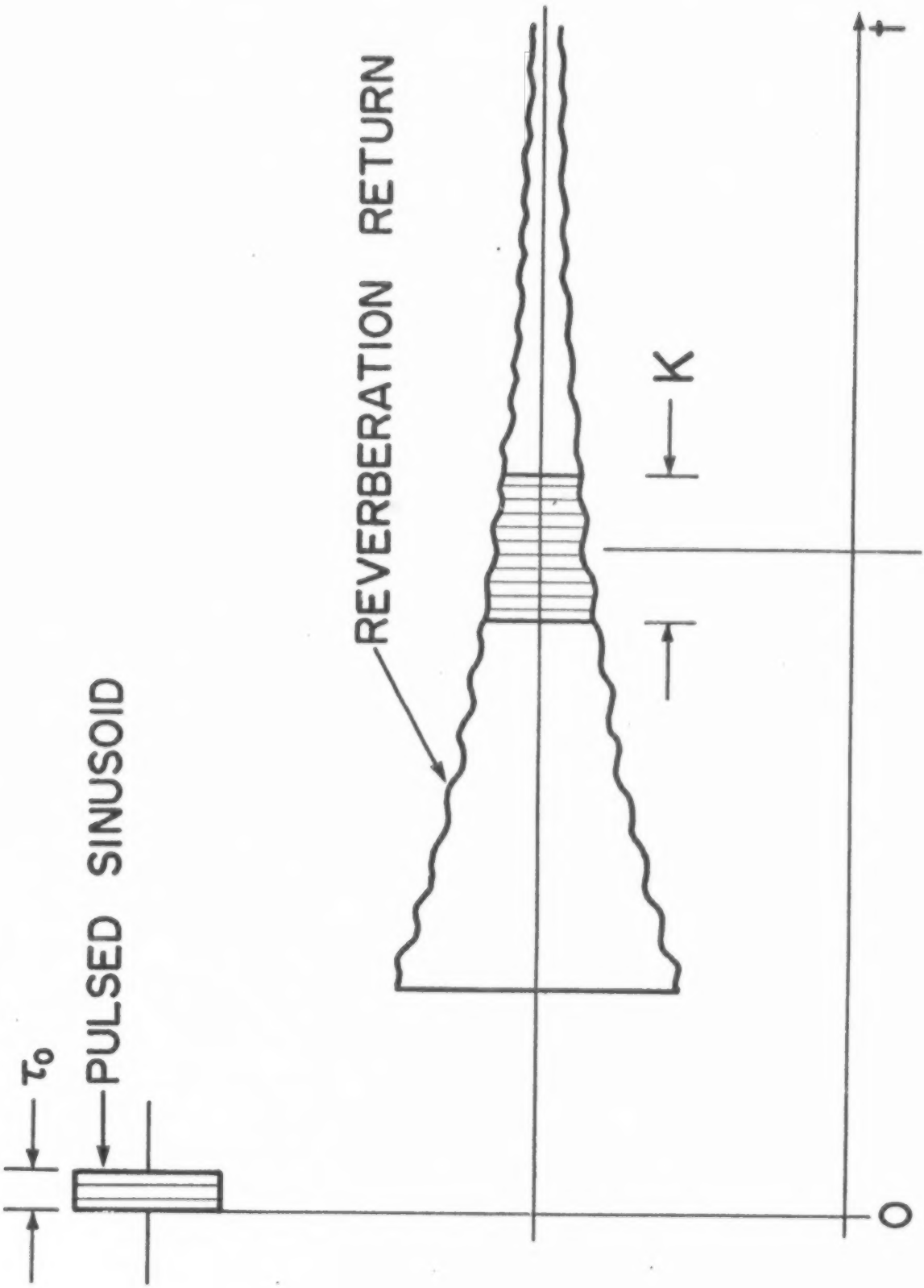

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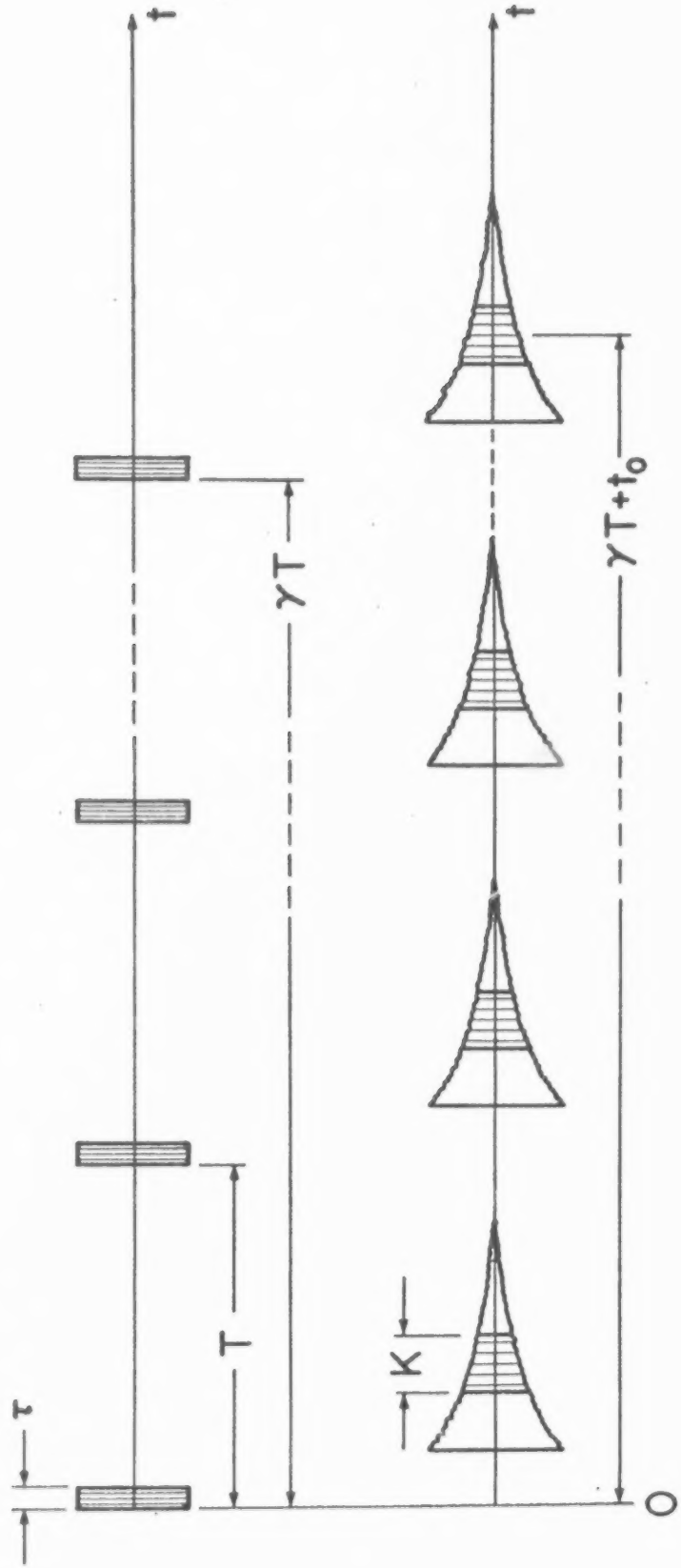

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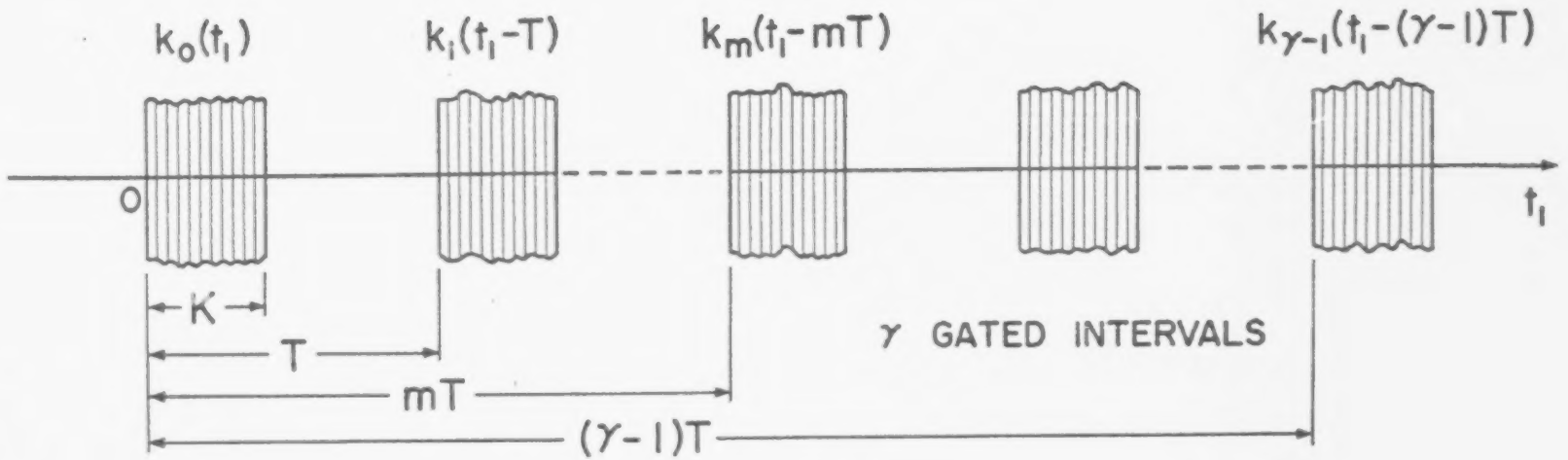

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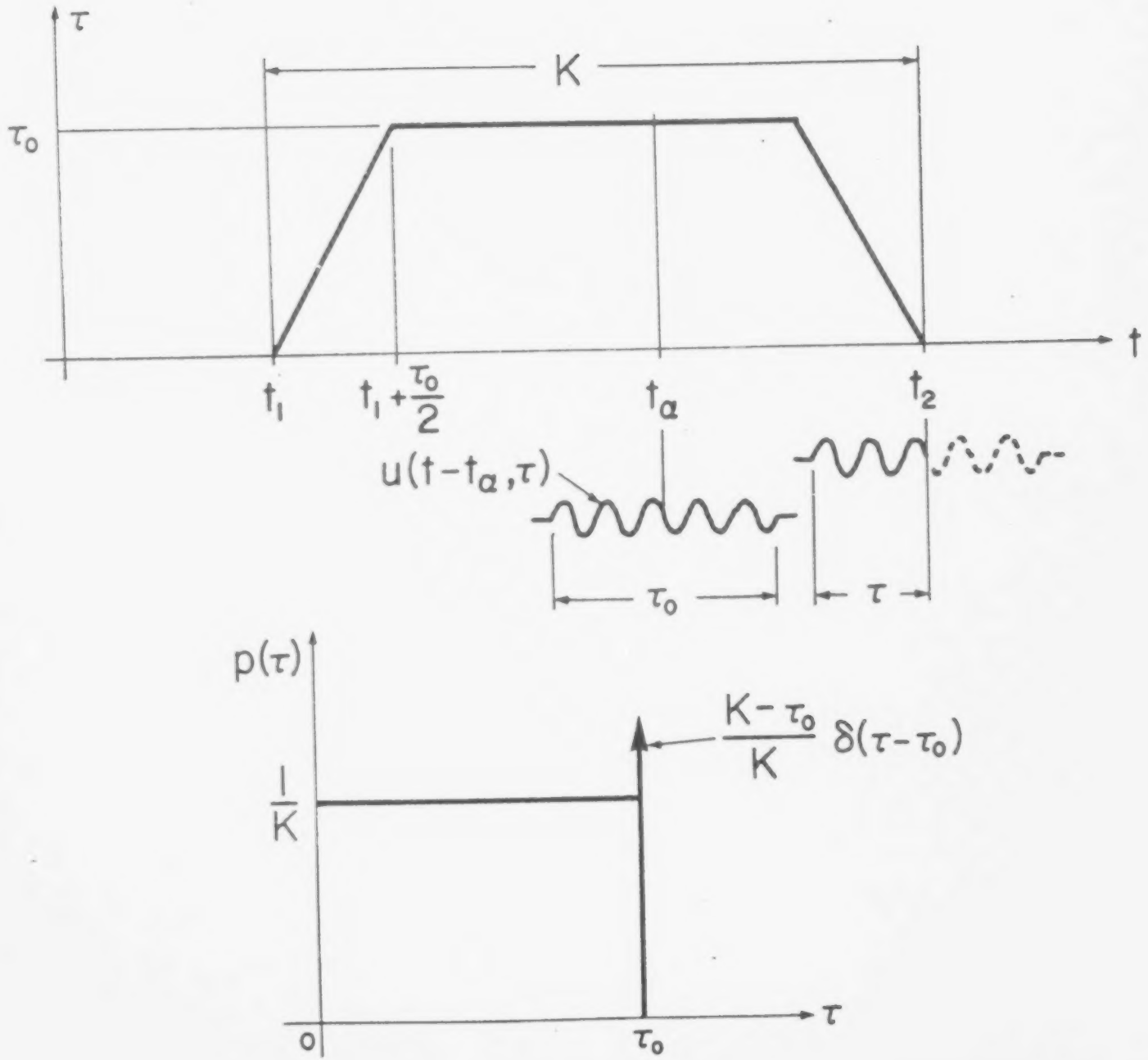

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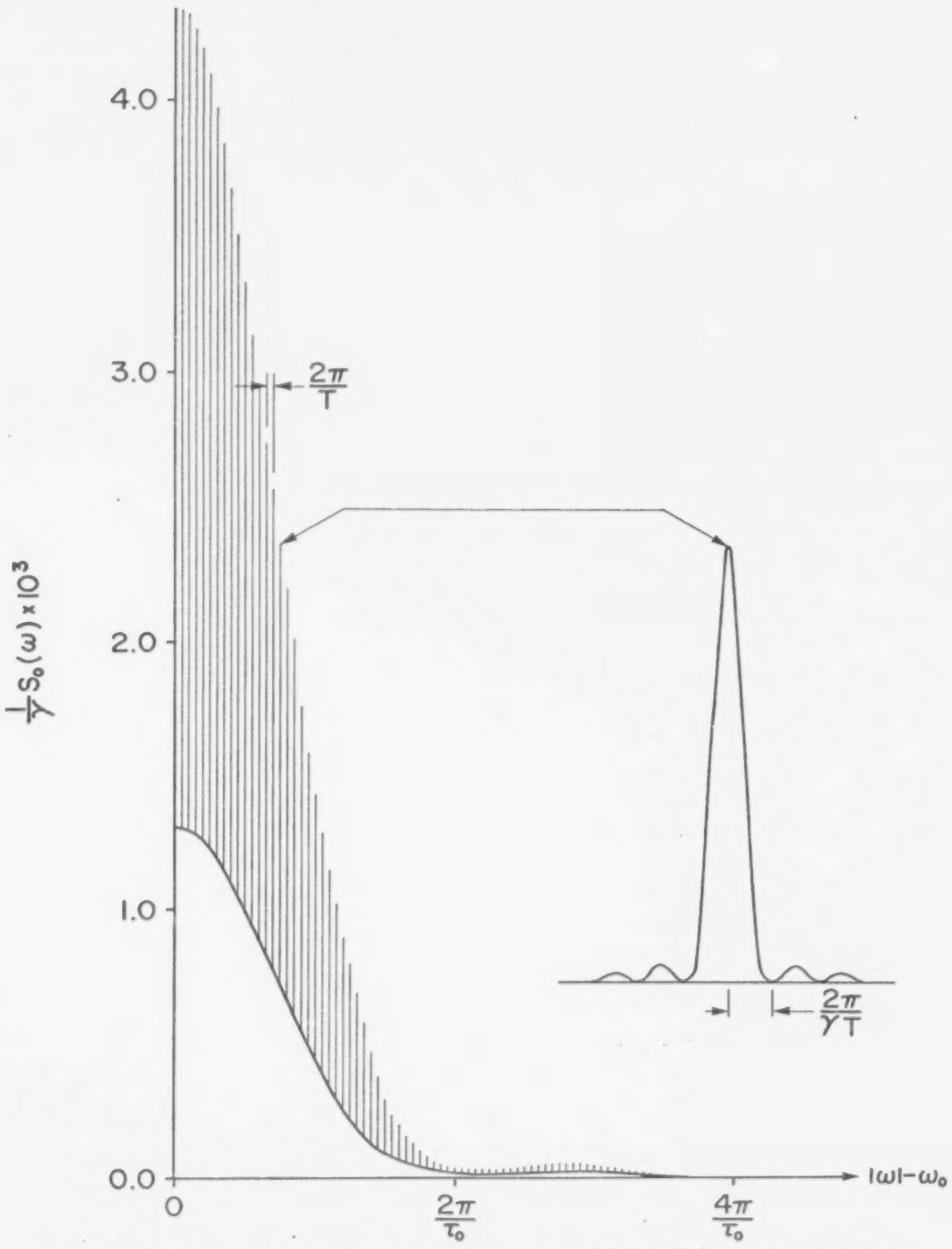

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UNDER-ICE ACOUSTIC STABILITY: PRELIMINARY EXPERIMENT AND THEORY OF MEASUREMENT (U)
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A B S T R A C T

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